On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations

C. Atkinson\textsuperscript{a}, E. Sarris\textsuperscript{b,*}, E. Gravanis\textsuperscript{b}, P. Papanastasiou\textsuperscript{c}

\textsuperscript{a}Department of Mathematics, Imperial College London, South Kensington Campus, London, UK
\textsuperscript{b}International Water Research Center (IWRC) NIREAS, University of Cyprus, 1678 Nicosia, Cyprus
\textsuperscript{c}Department of Civil and Environmental Engineering, University of Cyprus, 1678 Nicosia, Cyprus

\section{A R T I C L E I N F O}

Article history:
Received 27 May 2014
Received in revised form 21 April 2015
Accepted 13 May 2015
Available online xxxx

Keywords:
Wellbore recharge
Singular integral equations
Anisotropic formations

\section{A B S T R A C T}

The problem of determining the steady state pressure field for single and multi-well configurations with non-trivial wellbore boundary conditions is considered in this work as an integral equation problem. The aquifer, where the well configuration resides, is assumed to have different vertical and horizontal intrinsic permeabilities and it is bounded above and below by impermeable geological settings. The solutions of the integral equation, known as density functions, are studied from two points of view. First, the singular behavior of the density function is investigated by studying the singular part of the kernel of the integral equation; on this basis the density function is suitably expressed in terms of a non-singular counterpart, the reduced density function, for which a polynomial approximation is formulated and constructed numerically. The convergence of the approximation is studied with respect to the order of the polynomial and shown to be adequately fast. Second, the density functions for large depth to radius ratio and/or large horizontal to vertical permeability ratio exhibit similarity. The density functions depend on the parameters of the single-well problem through a single similarity parameter $c$. For large values of $c$, corresponding to the physical limits just mentioned, the density functions are reduced to essentially a single function, modulo a factor $(\log c)^{-1}$. This property simplifies considerably the analysis of all the large $c$ cases. Considering the case of two wells, as an illustrative example, we also show that the properties of the single well case are also exhibited by the multi-well density functions.

\section{1. Introduction}

Groundwater use is of fundamental importance to meet the rapidly expanding urban, industrial and agricultural water requirements particularly in semi-arid climates. Recently, the water resources management has also attracted considerable concern of scientists and engineers as it ensures availability around the year of groundwater for the various usages in semi-arid regions [1]. Recharge is defined in general sense as the downward flow of water reaching the water table, forming an addition to the groundwater reservoirs (i.e. aquifers) [2]. The principle recharge mechanisms were originally defined by Learner et al. [3] and are as follows: (1) diffuse percolation, as either an unsaturated flux or a saturated front (piston type flow), (2) macro-pore flow through root channels, desiccation cracks and fissures, (3) preferential flow caused by unstable wetting fronts and differentiated soil physical characteristics within the soil.

\* Corresponding author.

http://dx.doi.org/10.1016/j.apm.2015.05.010
0307-904X/© 2015 Elsevier Inc. All rights reserved.

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
Modeling the recharge process is thus a prerequisite for efficient and sustainable groundwater resource management in dry areas. Analytical models are often developed for better modeling of the physical processes involved in both surface and wellbore recharge [2]. In wellbore recharge, analytical models account for the effect of a single wellbore efficiency and wellbore geometry (radius, length) on the groundwater flow. Furthermore, models are utilized to deal with some groundwater wellbore recharge issues which include: (a) the finite well radius, (b) wellbore storage, (c) well partial penetration and (d) the presence of skin [4].

There is a vast literature on calculating pseudo-steady flow (productivity/injectivity) from wells especially for petroleum related applications. Much of this is concerned with the pressure diffusion equation which is applicable to laminar flow in porous medium deduced from using the Darcy law [5]. To study the flow characteristics various approximations have been made when applying differential equations to problems solving flow to/from wells. Often a well is approximated as a line sink and this is sufficient for many practical purposes (but, see for example [6]). In these works the complex physical boundaries induce boundary conditions which do not admit a straightforward analytical treatment and the respective problems are formulated as suitable integral equations, which are solved numerically [7]. In general, integral equation methods possess certain a priori advantages: the unknown function is defined only on the boundary of the domain, complex physical boundaries are easy to incorporate, the ill-conditioning associated with discretizing the governing equations is avoided, high-order accuracy is easy to attain and far-field boundary conditions are handled naturally.

Reducing the problem to an integral equation introduces the concept of a density function, the solution of the integral equation, which physically is interpreted as a point, or a distributed source that generates the assumed boundary conditions [6]. The density functions are mathematical objects which are interesting in their own right. Their determination most usually requires effective approximate/numerical methods. These methods may be improved considerably by exploiting fundamental properties of the density functions following from the structure of the kernel of the integral equation they obey. Additionally, as the density functions can always be given in a dimensionless form, studying their dependence on the similarity parameters of the problem may lead to interesting and possibly useful observations, especially regarding the behavior of the density functions near the limiting or critical values of the similarity parameters.

In this work we study the density functions from this point of view considering a setting which is as simple as it can be without being entirely trivial (trivial would be a setting such that the hydraulic potential along the wellbore does not depend on the vertical coordinates). We model the well as a cylinder of given length and radius oriented along the vertical principal axis of the permeability tensor. Thus we keep our geometry simple. We also work under steady state conditions. The boundary conditions imposed on the hydraulic potential along the wellbore boundary are chosen to be Dirichlet and dependent on the vertical position along the boundary. The dependence of the potential, or essentially equivalently of the injection rate, on the vertical position along the wellbore is not usual in hydrologic recharge. We have borrowed it from petroleum geomechanics where, while fracturing a formation from deep wells, a technique called multistage fracturing is used. With this technique, parts of the wellbore are isolated with packers in order to fracture the formation at the desired length of the deep well [8]. Apart from its physical interpretation this boundary condition serves mostly as a mathematically attractive choice as, in particular, we shall assume that the hydraulic potential along the wellbore boundary is a polynomial function of the vertical coordinate. The problem is reduced to a single-variable integral equation with the polynomial boundary condition as the source term.

The kernel of the integral equation possesses a logarithmic singularity. This is also true for a much more general class of related problems (see e.g. [7]). For the general theory of singular integral equations see e.g. [9]. In general, the presence of singularities makes the numerical solution of an integral equation very time-consuming and it is also rather difficult to achieve accurate results. There exist efficient methods for dealing with the problem of singular kernels, such as the hybrid Gauss-trapezoidal quadrature rules [10]. Nonetheless, the type of the singularity arising in wellbore recharge problems provides the resolution of the difficulty induced by the very presence of the singularity. The logarithmic kernel leads to soluble integral equations. Thus the singular behavior of the density function can be analyzed. We shall show that indeed, after factoring out a suitable singular function associated with the logarithmic kernel, what remains of the density function is a well behaved non-singular function. (Compare with [7]).

The method used in order to determine the claimed-to-be non-singular part of the density function, which we call the reduced density function, is to approximate it with polynomials of increasing order. One of our aims is to investigate how fast the polynomial approximations of the reduced density functions converge to a single well-behaved function. Clearly, the dual purpose of such an investigation is both to establish the approximate method as such and also to answer the theoretical question whether the reduced density function is indeed non-singular as claimed. We shall show that the sought for convergence is adequately fast. From the practical point of view, such results suggest that, most possibly, if the singular part of the kernel leads to a soluble integral equation then the polynomial approximation can be used as an effective tool in producing accurate solutions in that particular problem.

In the simple setting used in this work the density functions depend on the parameters of the problem essentially only through a single dimensionless parameter c, defined in Eq. (4). The similarity parameter c is interpreted as the apparent half-depth of the well in coordinates which are suitably defined so that the medium will appear isotropic, i.e. defined through suitable rescaling of the physical coordinates by the intrinsic permeability. When the well is deep and/or the horizontal permeability of the medium is much larger than the vertical then c is large. The behavior of the density for large c is an attractive mathematical question which is also of physical interest. In fairly usual situations, where also there is an assumed excess of horizontal permeability relatively to the vertical, the number c may easily be in the vicinity of 100, which
as we shall see qualifies as a 'large \( c \)' in the following sense. For values of the similarity parameter of that order the density functions exhibit similarity i.e. they are essentially reduced to a single one-variable function, modulo a factor \( (\log c)^{-1} \). From a practical point of view the similarity simplifies the analysis of the all the large \( c \) cases. Thus, one needs only to determine the density function for a certain large \( c \).

The structure of this paper is as follows: in Section 2, we construct the single well integral equation for the recharge problem and deduce the singular behavior of the density function. Section 3 describes the formulation of the numerical solution of the integral equation via the polynomial approximation of the density function, while in Section 4 we extend the methodology to account for the well interaction in multiwell arrangements. Section 5, presents the results claimed above for both the single well and the well interaction cases.

2. Formulation of the problem

2.1. The single well integral equation

We shall treat first the single well problem. Fig. 1 shows the graphical representation of the problem under consideration in Cartesian coordinates. We consider a vertical cylindrical wellbore of radius \( r_1 \) in a homogeneous medium which is bounded above and below by impermeable geological settings. We assume anisotropy between the horizontal and vertical direction. The horizontal directions \( X, Y \) are assumed as isotropic. Let \( X, Y, Z \) be the physical coordinates, denoted collectively by \( X_i \). The formulation is based on steady state conditions. The equation for the hydraulic potential \( \Phi \) follows from the Darcy velocity \( q_i \) given by (see e.g. [11]):

\[
q_i = -\frac{1}{\mu} k_{ij} \frac{\partial \Phi}{\partial X_j}, \quad \Phi = p + \gamma Z, \quad k_{ij} = \begin{pmatrix} k_h & 0 & 0 \\ 0 & k_h & 0 \\ 0 & 0 & k_v \end{pmatrix},
\]

where \( k_{ij} \) is the matrix of intrinsic permeability of the medium, \( \mu \) is the dynamic viscosity of the water, \( p \) is the pore pressure and \( \gamma \) the specific weight of the water. Everywhere outside the wellbore holds that:

\[
\frac{\partial q_i}{\partial X_i} = 0 \quad \text{i.e.} \quad k_{ij} \frac{\partial^2 \Phi}{\partial X_i \partial X_j} = 0.
\]

We define the isotropic coordinates \( x, y, z \) by:

\[
x = \frac{X}{r_1}, \quad y = \frac{Y}{r_1}, \quad z = \frac{Z}{r_1} \sqrt{k_h/k_v}.
\]

The isotropic coordinates are also dimensionless. The radial variable \( r = \sqrt{x^2 + y^2} \) expresses the distance from the center of the wellbore in units of the wellbore radius. (Note that often in what follows we will denote the position vector on the \( x, y \) plane by \( r \) i.e. \( r = (x, y) \)). Therefore in the isotropic coordinates (Eq. (3)), the wellbore has unit radius. Also its length is parameterized as \(-c \leq z \leq c\) where:

\[
c = \frac{d/2}{r_1} \sqrt{k_h/k_v},
\]

where \( d \) is the wellbore depth. The parameter \( c \) is the similarity parameter in this problem. It can also be viewed as the dimensionless half-depth of the well. Note that, the more permeable is the medium in the horizontal direction the more
the well appears longer in the isotropic coordinates. Therefore, the apparent length of the well in the isotropic coordinates expresses the vertical-horizontal anisotropy of the medium.

In the isotropic coordinates the hydraulic potential satisfies the boundary value problem with the following set of Eqs. (5a)–(5d):

\[ \nabla^2 \Phi = 0, \quad r > 1, \quad -c \leq z \leq c, \]

\[ \lim_{r \to 1^+} \Phi(r, z) = b(z), \]

\[ \lim_{r \to 0} \Phi(r, z) = 0, \]

\[ \partial_z \Phi(r, c) = 0 = \partial_z \Phi(r, -c) \]

for some function \( b(z) \) which we shall take to be a polynomial. This function \( b(z) \) has dimensions of pressure. The medium has been assumed bounded above and below by impermeable layers (Fig. 1). This fact is introduced in the problem through the Neumann boundary conditions (5d). We also introduce a boundary condition `at infinity` i.e. there is a potentiometric surface at some adequately large distance \( R_\infty \), which is chosen as the reference datum where the elevation is set to zero.

The Neumann boundary conditions in the \( z \)-direction impose that there is a solution which is independent of \( z \). In an integral representation of the solution \( \Phi \), this part of the solution needs to be isolated. Explicitly, this part is the vertical mean \( \Phi(r) \) of the field \( \Phi \) that satisfies the radial boundary conditions. Indeed, by the condition (5c) one finds that \( \Phi(r) \) satisfies the Laplace equation:

\[ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \Phi(r) = 0, \]

which implies \( \Phi(r) = C_0 + D_1 \log r \). The integration constants \( C_0 \) and \( D_1 \) are determined through the boundary conditions. By the condition (5d) one finds:

\[ \Phi(r) = C_0 \left( 1 - \frac{\log r}{\log R_\infty} \right), \quad C_0 \equiv \frac{1}{2c} \int_{-c}^{c} b(z)dz. \]

The integral representation of the non-trivial (\( z \)-dependent) part of the solution is built as follows. We introduce first the Neumann Green’s function \( g_{i}(z, z_1) \) which satisfies the boundary conditions and \( \partial_z g_i(c, z_1) = 0 = \partial_z g_i(-c, z_1) \) and the differential equation:

\[ \frac{\partial^2}{\partial z^2} g_i(z, z_1) - \frac{1}{c^2} g_i(z, z_1) = -2i \delta(z - z_1) + \frac{1}{c}, \]

where \( \delta \) is the Dirac delta function. The additive constant \( \frac{1}{c} \) in the equation subtracts the zero-mode (\( z \)-independent solution of (Eq. (10))) from the definition of the (Neumann) Green’s function. As a result the Green’s function satisfies the condition:

\[ \int_{-c}^{c} g_i(z, z_1) \, dz = 0. \]

In Appendix A we show that the following function is a solution of the Laplace equation for \( r \neq r_1 \):

\[ G(z, r; z_1, r_1) = \int_{0}^{\infty} dl \, g_i(z, z_1) J_0 (l |r - r_1|), \]

where \( r \) and \( r_1 \) are the position vectors of two points in the \( x, y \) plane i.e. \( r = (x, y) \) and \( r_1 = (x_1, y_1) \) and \( J_0(x) \) is the zeroth order Bessel function.

The integral representation of the solution amounts to forming a superposition of such solutions by introducing a function \( f(z_1, \theta_1) \) over the boundary of the wellbore, the density function, and write the solution as:

\[ \Phi(r, z) = \Phi(r) + \int_{-c}^{c} \int_{0}^{2\pi} G(r, r_1; z, z_1) f(z_1, \theta_1) d\theta_1 dz_1, \]

where the vector \( r_1 = (\cos \theta_1, \sin \theta_1) \) parameterizes the circumference of the wellbore. The function \( f \) has the dimensions of pressure and the function \( G \) is dimensionless.

The expression can, though, immediately be simplified because the problem of a single well possesses cylindrical symmetry and the density function does not depend on \( \theta_1 \). The angular integration is merely reduced to the calculation of the integral

\[ \int_{0}^{2\pi} J_0 (l |r - r_1|) d\theta_1. \]
By the Bessel function addition theorem [12]:
\[ J_0(|r - r_1|) = \sum_{m=\infty}^{m=\infty} e^{im(\theta - \theta')} J_m(|r|) J_m(|l|), \]
(13)

(\text{where } i \text{ is the imaginary unit}) we have that this integral can be calculated explicitly
\[ \int_{0}^{2\pi} J_0(|r - r_1|) d\theta = 2\pi J_0(|r|) J_0(|l|). \]

The representation (11) takes then the form
\[ \Phi(r, z) = \Phi(r) + \int_{-\infty}^{\infty} \left\{ 2\pi \int_{0}^{\infty} dl g_i(z, z_1) J_0(|l|) J_0(|l|) f(z_1) dz_1. \right\}
\]
(15)

The function \( \Phi \) as defined satisfies the Laplace equation for all \( r > 1 \). The boundary condition for \( r \rightarrow 1^+ \), (Eq. (5b)), leads to an integral equation for the unknown density function. This integral equation reads
\[ \int_{-\infty}^{\infty} G(z, z_1) f(z_1) dz_1 = b(z) - C_0, \quad G(z, z_1) \equiv 2\pi \int_{0}^{\infty} dl J_0^2(|l|) g_i(z, z_1), \]

where, we introduce a new function, \( G(z, z_1) \), the kernel of the fundamental integral equation. Solving Eq. (8) under the associated boundary conditions (Neumann) we find that \( g_i(z, z_1) \) is given by:
\[ g_i(z, z_1) = \begin{cases} 2 \cosh(l(c - z)) \cosh(l(c + z_1)) \frac{1}{\sinh(2lc)} - \frac{1}{l}, & z > z_1, \\ 2 \cosh(l(c - z_1)) \cosh(l(c + z)) \frac{1}{\sinh(2lc)} - \frac{1}{l}, & z < z_1. \end{cases} \]

(17)

This completes the explicit formulation of the problem at hand. We shall frequently refer to (16) as the fundamental integral equation.

2.2. Parity properties of the kernel

The Green’s function equation implies that:
\[ g_i(-z, -z_1) = g_i(z, z_1). \]

(18a)

Thus
\[ g_i(-z, -z_1) = g_i(z, -z_1). \]

(18b)

Subsequently the kernel of the integral equation satisfies
\[ G(-z, z_1) = G(z, -z_1). \]

(19)

This is important because the parity properties of the function \( f(z) \) are related to the those of \( b(z) - C_0 \). The even and odd part of the density function \( f(z) \) may generate only the even and odd part of \( b(z) - C_0 \) respectively. The linearity of the integral equation allows one to exploit this property and break the problem up into its even and odd parts as it will be described next. Let \( b_+ (z) \) and \( b_- (z) \) be the even and the odd part of the function \( b(z) \). Then the fundamental integral equation can be re-written as the pair of integral equations:
\[ \int_{-\infty}^{\infty} G(z, z_1) f_+(z_1) dz_1 = b_+(z), \]

(20a)

\[ \int_{-\infty}^{\infty} G(z, z_1) f_-(z_1) dz_1 = b_-(z) - C_0, \]

(20b)

where \( f_+(z) \) and \( f_-(z) \) are the even and odd parts of the density function, respectively. This greatly facilitates the numerical solution of the problem and we shall use it explicitly when calculating the unknown density functions \( f(z) \) in Section 5.

2.3. The singularity of the kernel and the density function

The kernel \( G(z, z_1) \) defined in Eq. (16) is logarithmically singular when \( z \) and \( z_1 \) are close. Clearly this behavior comes from the large \( l \) contribution to the defining integral of the kernel. Indeed, for large \( l \) the Green’s function \( g_i(z, z_1) \) can be written as
\[ g_i(z, z_1) \sim \begin{cases} 4 \exp(-2lc) \cosh(l(c - z)) \cosh(l(c + z_1)), & z > z_1, \\ 4 \exp(-2lc) \cosh(l(c - z_1)) \cosh(l(c + z)), & z < z_1, \end{cases} \]

(21)
This gives the following form to the kernel

\[
G(z, z_1) = \frac{4}{\lvert z - z_1 \rvert} K\left( -\frac{4}{\lvert z - z_1 \rvert^2} \right) + \frac{4}{2c - z - z_1} K\left( -\frac{4}{(2c - z - z_1)^2} \right) + \frac{4}{2c + z + z_1} K\left( -\frac{4}{(2c + z + z_1)^2} \right) 
\]

\[
+ \frac{4}{4c - \lvert z - z_1 \rvert} K\left( -\frac{4}{4c - \lvert z - z_1 \rvert^2} \right),
\]

(22)

where \(K(x)\) is the complete elliptic integral of the first kind. Its definition and certain properties of this function are given in the Appendix B. Note that this expression respects the Neumann boundary conditions at \(z = \pm c\). Consider first small separations, \(\lvert z - z_1 \rvert \to 0\), in the interior of the interval \(-c \leq z \leq c\), i.e. \(z\) and \(z_1\) are not close to the boundary points of the interval. Then, in the small separations limit, the behavior of the kernel is governed only by the first term in (22). Using the formula (B.4) we find

\[
G(z, z_1) \sim -2 \log \left( \frac{\lvert z - z_1 \rvert}{8} \right).
\]

(23)

The logarithmic kernel gives rise to a standard problem in the theory of integral equations. Differentiating the integral equation we obtain:

\[
-2 \int_{-c}^{c} f(z_1) \frac{dz_1}{z - z_1} = b'(z),
\]

(24)

where the prime denotes derivative. The solution to (24) depends on the assumed behavior of the solution near the boundaries \(z = \pm c\). The most singular solution to this equation can be written as [13]:

\[
-2\pi f(z) = -\frac{1}{\pi} \frac{1}{\sqrt{c^2 - z^2}} \left[ C + \int_{-c}^{c} \sqrt{c^2 - t^2} b'(t) \frac{dt}{z - t} \right].
\]

(25)

\(C\) is an arbitrary constant. We have mentioned that \(b(z)\) shall be assumed to be a polynomial. One may explicitly show each monomial \(z^k\) of \(b(z)\) produces polynomials of order \(k + 1\). This is shown in Appendix C. Therefore, then the quantity in the brackets is a polynomial (of the same order). Thus the asymptotic form of the kernel \(G(z, z_1)\) for small separations in the interior of the interval \(-c \leq z \leq c\) suggests the following general form for the density function:

\[
f(z) = -\frac{cF(z)}{\sqrt{c^2 - z^2}},
\]

(26)

where \(F(z)\) is a non-singular function (with dimensions of pressure) which we shall call as the reduced density function. A non-singular function can be approximated by polynomials. A priori, at the level of the exact integral equations (20), the relation (26) is a mere change of variable from \(f\) to \(F\). Following the lead coming from the previous analysis we shall look for polynomial approximations of the (exact) reduced density function \(F(z)\).

We turn now to the study the behavior of the kernel for small separations near to the boundary points of the interval \(-c \leq z \leq c\). We must make sure that the singular function \((c^2 - z^2)^{-1/2}\) which we have factored out by (26) is at least as singular as the actual behavior of the density function \(f(z)\) near the boundary points. Otherwise the reduced density function \(F(z)\) would be itself singular. Indeed, we argue below that \(f(z)\) diverges only logarithmically near the boundary points i.e. it exhibits a milder singular behavior.

Consider small separations in the vicinity of the boundary point \(z = -c\). That is, we take \(\lvert z - z_1 \rvert \to 0\) and also \(2c + z + z_1 \to 0\). The behavior of the kernel is governed now by the first term as well as the third term of (22). Using formula (B.4) we have

\[
G(z, z_1) \sim -2 \log \left( \frac{\lvert z - z_1 \rvert}{8} \right) - 2 \log \left( \frac{2c + z + z_1}{8} \right),
\]

(27)

which leads to the integral equation

\[
-2 \int_{-c}^{c} f(z_1) \left( \frac{1}{z - z_1} + \frac{1}{2c + z + z_1} \right) dz_1 = b'(z)
\]

(28)

in place of Eq. (24). Define variables \(t, t_1\) by

\[
z + c = 2c\sqrt{t}, \quad z_1 + c = 2c\sqrt{t_1},
\]

(29)

The boundary point \(z = -c\) is reached for vanishing \(t\). Eq. (28) becomes

\[
-2 \int_{0}^{1} f(z_1) \frac{\sqrt{t_1}}{t - t_1} dt_1 = \frac{b'(z)}{\sqrt{t}}.
\]

(30)
where the dependence of $z$ and $z_1$ on $t$ and $t_1$ is left understood. This equation can be solved by a formula analogous to Eq. (25) for the interval $[0,1]$. The solution is

$$-2\pi f(z) = -\frac{1}{\pi} \frac{1}{\sqrt{1-t}} \left[ C + \int_0^1 \frac{b'(2\sqrt{1-s}-c)}{t-s} \, ds \right].$$

Near the boundary point $z = -c$ (i.e. $t = 0$) the quantity of importance is the integral

$$\int_0^1 \frac{\sqrt{1-s}}{t-s} \, ds$$

as the $b$-dependent part of the integrand becomes constant near the boundary points for every monomial $b(z) = z^k/c^k$. This integral, which is explicitly defined by its Cauchy principal value, can be written as

$$\lim_{\varepsilon \to 0} \left\{ \int_0^1 \frac{\sqrt{1-t+s}}{s} \, ds - \int_0^1 \frac{\sqrt{1-t-s}}{s} \, ds \right\} \sim \log \frac{t}{c} + \log \varepsilon = \log t$$

for $t$ near 0. Therefore the density function $f$ is logarithmically singular for any polynomial $b(z)$, as promised. An analogous analysis can be given for the boundary point $z = \pm c$. Hence the reduced density function introduced by the relation (26) should be a regular function which tends to vanish as we approach the boundary points $z = \pm c$. This behavior seems to be indeed borne out by the numerical results shown below.

Finally, the change of variable $z = \cos \theta$, $0 \leq \theta \leq \pi$ facilitates calculations in both the exact and asymptotic integral equations. Then, presumably, the denominator in the definition (26) becomes part of the natural integration measure

$$\frac{dz}{\sqrt{c^2 - z^2}} = d\theta.$$  

Introducing the even $F_+(z)$ and odd $F_-(z)$ part respectively of the reduced density function $F(z)$, the fundamental pair of integral equations (20a) and (20b) now take the form:

$$c \int_0^\pi G(c \cos \theta, c \cos \phi) F_+(c \cos \phi) \, d\phi = b_+(c \cos \theta),$$  

$$c \int_0^\pi G(c \cos \theta, c \cos \phi) F_-(c \cos \phi) \, d\phi = b_+(c \cos \theta) - C_0,$$

(setting $z_1 = c \cos \phi$) also and these two Eqs. (35a) and (35b), are to be solved for the unknown even and odd parts of the reduced density function $F(z)$, i.e. the density functions $F_+(z)$.

3. Formulation of the numerical solution - The polynomial approximation of the density function

3.1. The polynomial approximation

It is adequate, as we shall explicitly show, to determine polynomial approximations of the reduced density function and therefore of its parts $F_+(z)$ as:

$$F_+(z) = F_1^+ + F_2^+ \frac{z^2}{c^2} + F_3^+ \frac{z^4}{c^4} + \cdots = \sum_{j=1}^{n_+} F_j^+ \left( \frac{z}{c} \right)^{2j-2},$$  

$$F_-(z) = F_1^- \frac{z}{c} + F_2^- \frac{z^3}{c^2} + F_3^- \frac{z^5}{c^4} + \cdots = \sum_{j=1}^{n_-} F_j^- \left( \frac{z}{c} \right)^{2j-1}.$$  

That is, one must determine the coefficients $F_j^+, j = 1, 2, 3, \ldots n_+$. This can be done with the following methodology. In order to determine a polynomial expression of $F_+(z)$ with $n_+$ terms, we should choose $n_+$ suitable values of the angle $\theta$: $\theta_i^+$, $i = 1, 2, 3, \ldots n_+$. We shall specifically choose:

$$\theta_i^+ = \frac{i}{n_+} \frac{\pi}{2}.$$  

Then we have to solve the two decoupled linear systems:

$$\sum_{j=1}^{n_+} M_{y_i}^+ F_j^+ = b_i^+$$  

$$\sum_{j=1}^{n_+} M_{y_i}^- F_j^- = b_i^-$$
where
\[
M_{ij}^+ = 2\pi c \int_0^\infty \int_0^\pi g_i(c \cos \theta_i, c \cos \phi) \cos^{2j-1} \phi \, d\phi, \quad b_i^+ = b_+(c \cos \theta_i),
\]
(39a)
\[
M_{ij}^- = 2\pi c \int_0^\infty \int_0^\pi g_i(c \cos \theta_i, c \cos \phi) \cos^{2j-2} \phi \, d\phi, \quad b_i^- = b_-(c \cos \theta_i) - C_0.
\]
(39b)
That is, the problem has been reduced to the calculation of the elements \(M_{ij}^\pm\) of the \(n_+ \times n_+\) matrices \(M^\pm\). The matrices \(M^\pm\) amount to a discrete version of the kernel of the integral equation in any given polynomial approximation of the reduced density function \(F(z)\). Note that the matrices \(M^\pm\) depend solely on the similarity parameter \(c\) of the problem.

### 3.2. Characteristic density functions

As discussed earlier in this work, we may write the polynomial \(b(z)\) as:
\[
b(z) = b_0 + b_1 z + b_2 z^2 + \cdots,
\]
where \(b_0, b_1, b_2, \ldots\) are quantities of the same dimension, and they all have the dimensions of pressure.

The linearity of the integral equation (16), or more specifically (35), allows us to determine the reduced density function \(F(z)\) associated with \(b(z)\) by determining the density functions associated with each monomial \(z^k/c^k\). Indeed, if by \(F^k(z)\) we denote the density function associated with the monomial \(z^k/c^k\) then the density function reads
\[
F(z) = b_1 F^1(z) + b_2 F^2(z) + \cdots.
\]
The functions \(F^k(z)\) associated with the monomial of order \(k\) will be called characteristic density functions. The functions \(F^k(z)\) are dimensionless. That means that \(F^k(z)\) depend only on the similarity parameter \(c\). The characteristic density functions are then the main objects one needs to study and in particular their dependence on the similarity parameter \(c\).

We shall study the polynomial approximations of the functions \(F^k(z)\) through method discussed in Section 3.1. We know from the study of the kernel singularity that these polynomial approximations must be of order \(k\) or higher. Moreover, the parity of the monomial \(z^k/c^k\) dictates the parity of \(F^k(z)\). If \(k\) is even or odd then \(F^k(z)\) is approximated by an even or odd polynomial, \(F^1(z)\) or \(F^2(z)\) respectively, of degree \(k\) or higher. Explicitly, for every monomial \(z^k/c^k\) with even \(k\), one has to determine \(n_+\) coefficients \(F^1_i\) with \(2n_+ - 1\) greater or equal to \(k\), and every monomial \(z^k/c^k\) with odd \(k\), one has to determine \(n_-\) coefficients \(F^2_i\) with \(2n_- - 1\) greater or equal to \(k\).

### 4. Interaction of multiple (distant) wells

In this section we will extend the methodology described in previous sections to account for the wellbore interaction. Let us consider an arrangement of \(N_w\) vertical wells, of the same radius \(r_1\), for simplicity, whose centers are located at \((x_i, y_i), i = 1, \ldots, N_w\). We shall work under the assumption that the distance between the wells is much larger than the radius of the wells. This allows us to introduce an approximation which is mathematically encoded into two conditions. First, the single-well boundary condition (5b) is replaced by the condition:
\[
\frac{1}{2\pi} \int_0^{2\pi} \Phi d\theta = b(z)
\]
for all \(I = 1, \ldots, N_w\). (We assume that all wellbores have the same boundary condition \(b(z)\). That can easily be generalized to different boundary conditions but we shall not need it.) The variable \(\theta_0\) is an angle that parameterizes the circumference of the wellbore. This condition expresses the fact that the hydraulic potential changes very little around the circumference of the wellbore for distant wells [11]. The second condition is that the \(N_w\) density functions \(f_i\) associated with the multi-well arrangement depend only on the vertical position \(z\) along the boundary of the wellbore and not on the angular position \(\theta_0\). Specifically, as the linearity of the Laplace equation allows us to write the hydraulic potential as a sum of one-well type of expressions, the potential can be written as:
\[
\Phi(x,y,z) = \Phi(x,y) + \sum_{i=1}^{N_w} \int_{-c}^{c} \left\{ \int_0^{2\pi} g_i(z, z_1) \left[ \int_0^{2\pi} f_i(l|\mathbf{r} - \mathbf{r}_l|) d\theta_l \right] \right\} f_i(z_1) dz_1,
\]
i.e. there is one term for each well that involves a density function that depends only on \(z\). One should note that the two conditions are strongly related: Under an exact treatment, the density functions would be two-variable functions \(f_i(z, \theta_0)\) and the boundary condition (41) would not suffice to determine the field \(\Phi\) in space. The vector \(\mathbf{r}_l = x_l + (\cos \theta_l, \sin \theta_l)\) parameterizes the points of the \(l\)th wellbore circumference and \(\mathbf{r}_d = (x_1, y_1)\) is the position of the center of the well on the \(x, y\) plane. Also we have:
\[ \Phi(x, y) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Phi(x, y, z) \, dz. \]  

(43)

Integrating out the angle variables \( \theta \) by using the identity (13) as before we obtain:

\[ \Phi(x, y, z) = \Phi(x, y) + \sum_{j=1}^{N_w} \int_{-\pi}^{\pi} \left\{ 2\pi \int_{0}^{\infty} dl \, g_j(z, z_1) J_0(|r - r_d|) |J_0(l)\right\} f_j(z_1) \, dz_1, \]  

(44)

\( \Phi(x, y) \) satisfies the Laplace equation:

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = 0 \]  

(45)

and the relevant solution is given from:

\[ \Phi(x, y) = C_{\text{multi-well}} \sum_{j=1}^{N_w} D_j \log |r - r_d|. \]  

(46)

At the \( j \)th wellbore, the position vector \( r \) in the argument of the logarithm is set equal to \( r = r_d + (\cos \theta_j, \sin \theta_j) \). Therefore, \( r - r_d = r_d - r_d + (\cos \theta_j, \sin \theta_j) \). Implementing the boundary condition (41) the following identity is useful:

\[ \int_{0}^{2\pi} \log |r + (\cos \theta, \sin \theta)| \frac{d\theta}{2\pi} = \log |r|, \]  

(47)

unless \( r = 0 \) in which case the result is clearly 0. Then the boundary condition (41) implies

\[ C_{\text{multi-well}} + \sum_{j=1}^{N_w} D_j \log R_{ij} = \frac{1}{2\pi} \int_{-\pi}^{\pi} b(z) \, dz = C_0, \]  

(48)

where \( R_{ij} = |r_d - r_d| = \sqrt{(x_i - x_j)^2 + (y_i - y_j)^2} \) is the distance between the centers of the wells labeled by \( i \) and \( j \). Note that, if the arrangement of the wells is symmetric then all the \( D_j \) coefficients are the same. The boundary condition ‘at infinity’ i.e. the condition (5c), which involves some large radius \( R_\infty \), gives

\[ C_{\text{multi-well}} + \left\{ \sum_{j=1}^{N_w} D_j \right\} \log R_\infty = 0. \]  

(49)

These conditions determine completely the constants \( C_{\text{multi-well}} \) and \( D_j \). The solution is now reduced to the linear system of \( N_w \) unknowns as:

\[ -D_j \log R_\infty + \sum_{j=1}^{N_w} D_j \log \frac{R_{ij}}{R_\infty} = C_0. \]  

(50)

We may now deal with the non-trivial part of the solution of Eq. (42) i.e. the part that involves the density functions. If the wells are infinitely distant then \( f_j(z) = f_{\text{single-well}}(z) \), which obeys Eq. (16). The interaction of distant wells lies in the deviation of \( f_j(z) \) from \( f_{\text{single-well}}(z) \). This means that we can write \( f_j(z) \) as:

\[ f_j(z) = f_{\text{single-well}}(z) + \delta f_j(z), \]  

(51)

where \( \delta f_j(z) \) is by definition a small quantity. We shall use this decomposition of the density functions to form perturbation series. The functions \( \delta f_j(z) \) essentially quantify the interaction of the distant wells.

We insert the previous decomposition of the density functions into the general expression (44) and impose the boundary condition (41). In simplifying the result, one needs the single well integral equation (16), the Eq. (48) deriving from the application of the boundary condition (41) on the \( \Phi \) part of the hydraulic potential, and the identity:

\[ \frac{1}{2\pi} \int_{0}^{2\pi} f_0(|r_d - r_d - (\cos \theta_j, \sin \theta_j)|) d\theta_j = f_0(R_{ij}) |J_0(l)|, \]  

(52)

which is merely one more application of the identity (14). One eventually finds:

\[ \int_{-\pi}^{\pi} \left\{ 2\pi \int_{0}^{\infty} \log g_j(z, z_1) J_0^2(l) \right\} f_{\text{single-well}}(z_1) \, dz_1 \]

\[ + \sum_{j=1}^{N_w} \int_{-\pi}^{\pi} \left\{ 2\pi \int_{0}^{\infty} \log g_j(z, z_1) J_0^2(l) \right\} \delta f_j(z_1) \, dz_1 = 0. \]  

(53)
This expression defines a perturbation expansion for $\delta f$, with $J_0(|R_0|)$ understood as the perturbation order parameter. The first order result for $\delta f$ is obtained by dropping the last term in the expression. The higher order corrections are derived by iterations. For the first order result, which is the result of interest, we obtain:

$$
\int_{-c}^{c} G(z,z_1) \delta f_j(z_1) dz_1 = \delta b_j(z)
$$

$$
\delta b_j(z) = -\sum_{l=1}^{N_w} \int_{-c}^{c} \left\{ 2\pi \int_{0}^{\infty} dl g_l(z,z_1) j^l_0(l|R_0|) \right\} f_{\text{single-well}}(z_1) dz_1
$$

where $G(z,z_1)$ is the kernel (9) of the fundamental integral equation (16). Once the single well density function $f_{\text{single-well}}(z)$ is calculated, the function $\delta b_j(z)$ can also be computed as well. (Note that only the factor $J_0(|R_0|)$ is actually summed in the expression (54) as no other quantity depends on the well number index $j$.) Therefore the problem again is reduced to the solution of the fundamental integral equation (16) and one may proceed as explained in Section 3.

5. Numerical solution of the characteristic density functions $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, $F^{(4)}$

As discussed in Sections 2.1 and 2.2 the density function $f(z)$ associated with the boundary condition $b(z)$ can be derived by determining its even and odd parts, $f_{\text{e}}(z)$, which are associated with the even and odd parts of $b(z)$, respectively. The same applies in the reduced density function $F(z)$, defined by Eq. (26). Thus, as discussed in Section 3, given that we are interested in a polynomial $b(z)$, we may determine and study separately the density functions associated with each monomial term of $b(z)$. We have called these functions as the characteristic density functions $F^{(k)}(z)$. The characteristic density functions $F^{(k)}(z)$, are associated with the $k$th power term of $b(z)$, specifically with the monomial $z^k/c^k$. We shall restrict our attention to the first four characteristic densities functions, i.e. those associated with the linear, the quadratic, the cubic and the quartic term of $b(z)$. One should bear in mind that the possible constant (zeroth order) term in $b(z)$ is associated with the $z$-independent part of the solution (Eq. (7)) and there is no characteristic density functions associated with it.

We shall be concerned with two matters. As discussed in Section 3, the functions $F^{(k)}(z)$ will be determined by constructing polynomial approximations given by Eqs. (36a) and (36b). The analysis of Section 2.3 suggests that $F^{(k)}(z)$ requires a priori – at least – order $k$ polynomial approximations, as such a polynomial is produced by the singular part of the kernel alone. The first issue of interest is then how fast the approximating polynomials converge to the exact function as the order of the polynomial increases. It turns out that the odd characteristic functions converge rather faster than the even ones. Also, as mentioned in Section 3.2 the functions $F^{(k)}(z)$ depend only on the dimensionless grouping $c$, the similarity factor of the problem defined in Eq. (4). The second issue of interest is then the behavior of $F^{(k)}(z)$ as $c$ changes and especially as $c$ increases to large values. Physically, the large values of $c$ are related to deep wells and large ratio of the horizontal to the vertical permeability. Both limits are of interest. Regarding this matter we shall argue, based on our numerical results, that the density functions exhibit similarity. Specifically, the density functions appear to converge to a single function for large values of $c$ once they have been scaled with $\log c$.

The polynomial approximations of the functions $F^{(1)}$, $F^{(2)}$, $F^{(3)}$, $F^{(4)}$ are shown in Figs. 2–5 for four different values of the similarity parameter (i.e. $c = 1, 5, 10, 50$). The approximating polynomials are conveniently plotted as functions of $z/c$. In the odd cases, i.e. in the case of the functions $F^{(1)}$ and $F^{(3)}$, the number $n$ labeling the curves in the plots is the number of terms in the approximating polynomial whose order is $2n - 1$ (see Eq. (36a)). In the even cases, $F^{(2)}$ and $F^{(4)}$, the approximating polynomial are of order is $2n - 2$ (see Eq. (36b)), where $n$ is again the number labeling the curves in the plot. Therefore in Figs. 2 and 4 the approximating polynomials of the functions $F^{(1)}$ and $F^{(3)}$ up to order 11 shown, while in Figs. 3 and 5 are shown the approximating polynomials of the functions $F^{(2)}$ and $F^{(4)}$ up to order 16.

Figs. 2 and 4 show that the approximating polynomials of the odd density functions converge fairly fast and in a monotonic manner to the exact function. On the other hand, Figs. 3 and 5 show that the approximating polynomials of the even density functions converge less fast to the exact function through diminishing up and downs, creating the relatively fuzzy picture shown in Figs. 3 and 5. Actually, the curves corresponding to the cases $n = 7, 8, 9$ are hard to distinguish visually. Additionally, the convergence in the even cases is somewhat subtle. The basic property of the Green’s function $g(z,z_1)$ expressed by Eq. (9) is inherited by the kernel $G(z,z_1)$. Through the fundamental equation (16) this property implies that the density functions are defined up to an arbitrary additive constant. Replacing $f(z)$ with $f(z) + \text{constant}$ leaves the integral equation (16) invariant. This fact is another illustration of the association of the density functions solely with the $z$-dependent part of the hydraulic potential $\Phi$. Clearly, this observation is important only for the even part of the density function. Recalling Eq. (26), this arbitrariness translates at the level of the reduced density function $F(z)$ as the replacement

$$
F(z) \rightarrow F(z) + \text{constant}\sqrt{c^2 - z^2},
$$

which leaves the integral equation (16) invariant and produces a completely equivalent reduced density function. The convergence shown in Figs. 3 and 5 is achieved only after subtracting from each even polynomial $F_{\text{e}}(z)$ the quantity $F_{\text{e}}(0)\sqrt{1-z^2/c^2}$, as a normalizing procedure that produces an equivalent density function. Finally, one may observe that

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
the curves in the $c = 50$ graph in every one of the Figs. 2–5 are much closer to one another than the curves for lower $c$, suggesting a faster convergence of the polynomial approximation when the similarity parameter $c$ is large.

Next we turn to the second issue of interest i.e. the behavior of the density functions under the variation of the similarity parameter $c$ and especially their behavior for large values of $c$. For this study we use the highest order polynomial

![Figure 2: Polynomial approximations of the characteristic density functions for the linear term of $b(z)$.](image2)

![Figure 3: Polynomial approximations of the characteristic density function for the quadratic term of $b(z)$.](image3)
approximation derived previously in each case. That is, we use the $n = 6$ polynomials in the case of the odd density functions $F^{(1)}, F^{(3)}$ and the $n = 9$ polynomials in the case of the even density functions $F^{(2)}, F^{(4)}$.

The results are depicted in Figs. 6 and 7. The similarity parameter $c$ is chosen to vary from $c = 5$ to $c = 120$. The Figs. 6(A), (C) and 7(A), (C) show the indicated density function multiplied by $\log c$. In Figs. 6(B), (D) and 7(B), (D) the peak density functions $F^{(1)}, F^{(3)}$ and $F^{(2)}, F^{(4)}$ are shown.

**Fig. 4.** Polynomial approximations of the characteristic density functions for the cubic term of $b(z)$.

**Fig. 5.** Polynomial approximations of the characteristic density functions for the quartic term of $b(z)$. 

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
of the scaled density functions is used as a convenient indicator of convergence. The convergence of the scaled density functions to a single function, as shown in Figs. 6(A), (C) and 7(A), (C), can be described as excellent arising in a clean, monotonous manner with increasing $c$. Perhaps more descriptively, the suggested convergence is also exhibited through the variation of the peak of the scaled density functions with $c$, shown in the Figs. 6(B), (D) and 7(B), (D). For the case of the function $F^3$ the convergence is essentially self-evident. For the higher $k$ density functions one observes that the rate of convergence slows down nonetheless the convergence is still rather evident.

Fig. 6. Large $c$ convergence of scaled characteristic density function $\log cF$ (odd cases).

Fig. 7. Large $c$ convergence of scaled characteristic density function $\log cF$ (even cases).
The suggested, or conjectured, convergence, which is observed numerically, is perhaps the most interesting result of this work. Specifically, if \( f^{(k)}(z) \) is a characteristic density function then \( \log c F^{(k)}(uc) \) approaches a well-defined function of the variable \( u \) for infinitely large \( c \).

6. Two well interaction

Consider the case of multiple wells on each of which we apply the non-trivial injection conditions \( b(z) \). For simplicity we assume that the wells are distant i.e. the distance between them is much larger than their radius. This is realistically quite meaningful. We will try to answer the following question. Do the density functions of the multi-well configurations share the properties found in the single well density functions? We are interested particularly in the property of similarity.

The formulas required to study this question were derived in Section 4. The density function of the well labeled by \( I \) can be written as \( f_I(z) = f_{\text{single-well}}(z) + \delta f_I(z) \). \( f_{\text{single-well}}(z) \) is the isolated (single) well density function associated with the injection conditions \( b(z) \) i.e., the function we constructed and studied in the previous Section 5; \( \delta f_I(z) \) is the perturbation of the single well density function due to the presence of the other wells. The question posed above concerns essentially the perturbation of the density functions associated with the two wells are identical, \( \delta f_I(z) = \delta f_J(z) \). We shall denote both these perturbations by a single symbol, \( \delta f(z) \). Let us re-write Eq. (54) explicitly for this configuration:

\[
\int_{-\infty}^{\infty} G(z, z_1) \delta f(z_1) dz_1 = \delta b(z)
\]

\[
\delta b(z) = - \int_{-\infty}^{\infty} \frac{2\pi}{c} \int_0^\infty d[l] g_c(z, z_1)f^0_0(l) f_0(I) f_0(R) f_{\text{single-well}}(z_1) dz_1
\]

where \( R \) denotes the distance, or spacing, between the two wells. The perturbation \( \delta f(z) \) depends solely on the similarity parameter \( c \) and the spacing \( R \). Recalling Eq. (3), one should bear in mind that \( R \) is dimensionless, expressing the distance between the wells relatively to the well radius \( r_i \).

We study the large \( c \) behavior of the perturbation \( \delta f(z) \) for four different values of spacing (i.e. \( R = 3c, 4c, 5c, 6c \)) for \( c \) ranging from \( c = 5 \) to \( c = 120 \) as in the single well examples. We consider the perturbation to the characteristic density functions \( F^{(1)} \) and \( F^{(2)} \) i.e. those associated with the linear and the quadratic term in the boundary condition \( b(z) \). Fig. 8(A)–(D) shows the scaled perturbation, i.e. the quantity \( \log c \delta F^{(1)}(z) \), as a function of \( z/c \) for the four different values of the spacing \( R \) respectively. Fig. 10(A)–(D) present the scaled perturbation \( \log c (-\delta F^{(2)}(z)) \) as a function of \( z/c \) for the same values of the spacing \( R \). As in the single well case, for large \( c \) one observes a clean and monotonic convergence of the scaled density functions to a single function. (Presumably, one should observe that the perturbation has always a different sign than the single well
density function i.e. it contributes always negatively in the full density function. Indeed, in the case of $F^{(1)}$ the perturbation is positive for the negative $z$ while the single well function is positive for the positive $z$, while in the case of $F^{(2)}$ the perturbation has simply the opposite sign. This means that the two-well configuration produces a lower hydraulic potential than the potential produced by algebraically adding up the contributions of two isolated wells, in the same positions.) Also, following our practice in the single well case, the peak of the scaled density functions is plotted as a function of $c$ as a more quantitative measure of the conjectured similarity. In Fig. 9(A)–(D) the peak of $\log c \delta F^{(1)}(z)$ for the four different values of spacing we

**Fig. 9.** The peak of the scaled perturbations $\log c \delta F^{(1)}$ plotted against the similarity parameter $c$ for well spacing $R$ equal to (A) $3c$, (B) $4c$, (C) $5c$, (D) $6c$.

**Fig. 10.** The scaled perturbations $\log c (-\delta F^{(2)})$ for well spacing $R$ equal to (A) $3c$, (B) $4c$, (C) $5c$, (D) $6c$. 

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
consider, $R = 3c$, $4c$, $5c$, $6c$, is plotted against $c$. In Fig. 11(A)–(D) the peak of log $c \delta F^{(2)}(z)$ is plotted against $c$ for the same values of well spacing. Here, a slower convergence is observed for large $c$ compared to the single well density functions, nonetheless the convergence can be regarded as evident. In fact, the scaled perturbation log $c \delta F^{(2)}(z)$ appears to converge faster than the scaled perturbation log $c \delta F^{(1)}(z)$, as opposed to the way the associated single well density functions behave, as seen by comparing Figs. 9 and 11 with Fig. 6(B) and (D). Finally, an obvious similarity is observed in the Figs. 9 and 11 for different values of spacing $R$. Up to an $R$-dependent factor, analogous to $(\log c)^{-1}$, the density functions appear entirely similar. The dependence, though, of this factor on $R$ does not seem to be as simple and elegant as it is in the case of the similarity parameter $c$.

7. Comparison of the proposed approach with a Fourier analytical solution

We present here an example of comparison between the results of the present work and the related exact solution of the given case. An exact solution is possible due to the simplicity of the problem we have chosen, which facilitates both the presentation of the method as well as its straightforward validation in checkable examples. We shall consider the case of the linear term $z/c$ of the function $h(z)$ as explained in Section 3.2. The associated exact solution is given in Appendix E as a Fourier series. The solution through the present method is associated with the characteristic function $F^{(1)}(z)$, determined as explained in Section 3 and the results presented in Section 5. The characteristic density function is turned to a density function through Eq. (26) and to a hydraulic by Eq. (15). We consider the case $c = 1$ and use the highest polynomial

Fig. 11. The peak of the scaled perturbations $\log c (-\delta F^{(2)})$ plotted against the similarity parameter $c$ for well spacing $R$ equal to (A) $3c$, (B) $4c$, (C) $5c$, (D) $6c$.

Fig. 12. Plot of the function $\Phi(r, z) - \Phi(r)$ versus $z/c$ for five different distances from the wellbore center.
approximation of $F^1(z)$ we constructed and presented and Section 5 (corresponding to $n_r = 6$, i.e. a polynomial of order $2 \times 6 - 1 = 11$), in order to achieve the highest precision results according to the method. We calculate the hydraulic potential through our method at the following points

$$r = j c, \quad z = c \cos \left[ \frac{i}{3T} \times \frac{\pi}{2} \right],$$

where $j = 1, 2, \ldots, 5$, is that is, for the five different case of distance $r$ from the wellbore center as multiple of the wellbore half-length, and $i = 1, 2, \ldots, 30$, that is, for thirty points in the vertical direction in the upper half of the wellbore ($0 < z < c$), as the lower in the lower half is obtained by symmetry ($b(z) = z/c$ and therefore the solution is an odd function of $z$).

The results are given in Fig. 12. We plot the function $\Phi(r, z) - \Phi(r)$, that is hydraulic potential minus the $z$-independent function $\Phi(r)$ given by in Eq. (7). The continuous lines are the graphs of Fourier series exact solution while the circles correspond to the results of the present numerical method. The agreement is indeed excellent.

8. Discussion and conclusions

This paper studies the solutions of integral equations, known as the density functions, arising in single and multi-well configurations in steady state conditions. The hydraulic potential is assumed to be a polynomial function of the vertical coordinate along the boundary of the wellbore. The aquifer is assumed to have different vertical and horizontal intrinsic permeabilities and it is bounded above and below by impermeable planes.

The kernel of the integral equation for the single-well problem, Eq. (16), is singular. This property largely shapes the density functions which become singular themselves at top and the bottom of the wellbore. The singular behavior can be identified by studying the singularity of the kernel and the integral equation associated with it. At this point it is not actually clear whether there are also subdominant singularities in the density function which cannot be unraveled by the presented analysis. On the other hand the analysis suggests that the density function can be factorized into a part which is adequately singular so that the other part, which we have called as the reduced density function, is, most possibly, non-singular. This is investigated by studying polynomial approximations of the reduced density functions, which are constructed numerically. The approximating polynomials appear to converge rather fast to a single non-singular function as the order of the polynomial increases. Also, in full consistency with the singularity analysis, the approximating polynomials exhibit a tendency to vanish near the top and the bottom of the wellbore, especially as the order of the polynomial increases. At the same time this investigation provides a test for the efficiency of the polynomial approximation as a practical method for determining the density function, and therefore the pressure field, with adequate accuracy.

The dependence of the density functions on the parameters of the problem is realized through the similarity parameter $c$, defined in Eq. (4). It can be interpreted as the apparent half-length of the well in isotropic coordinates, defined in Eq. (3). Large $c$ means large medium thickness to well-radius ratio and/or large horizontal to vertical permeability ratio. Large values of the number $c$ may arise quite easily in actual situations; hence this limit is of physical interest, apart from the obvious mathematical one. It is found that the density functions exhibit similarity for large $c$; in that limit the density functions appear to converge to a single function of $z/c$, where $z$ is the isotropic vertical coordinate, modulo a factor $(\log c)^{-1}$. From a practical point of view this property simplifies the analysis of the large $c$ cases. Based on this observation, one may conclude that similar simplifying properties may arise, in the associated limits, in much more complicated well recharge problems which one may encounter in practice.

Regarding multi-well configurations, we use the symmetric two-well configuration of distant wells as an illustrative example. We show that the mentioned similarity, which originated in the single-well problem, survives also in the problem of multiple distant wells.

Acknowledgments

This work was co-funded by the European Regional Development Fund and the Republic of Cyprus through the Research Promotion Foundation (Strategic Infrastructure Project ΝΕΑΥΠΟΔΟΜΗ/ΣΤΡΑΤΗ/0308/09).

Appendix A

The zeroth order Bessel function $J_0(x)$ satisfies the equation

$$\left[ \frac{1}{x} \frac{\partial}{\partial x} x \frac{\partial}{\partial x} + 1 \right] J_0(x) = 0. \tag{A.1}$$

The (Neumann) Green’s function satisfies Eq. (8). Multiplied by a factor $e^{-id}$ this equation reads

$$\frac{\partial^2}{\partial z^2} e^{-id} g_1(z, z_1) - \delta^2 e^{-id} g_1(z, z_1) = -2i e^{-id} \delta(z - z_1) + e^{-id} \frac{l}{c}, \tag{A.2}$$

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
for some small positive number $\varepsilon$.

The variables $r$ and $z$ are the radial and vertical coordinates respectively, of a cylindrical system $r$, $\phi$, $z$. It is straightforward to show that the function $g_1(z, z_1)J_0(lr)$ satisfies the equation

$$\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right) \left\{ e^{-i\varepsilon} g_1(z, z_1)J_0(lr) \right\} = -2l e^{-i\varepsilon} J_0(lr) \delta(z - z_1) + e^{-i\varepsilon} \frac{l}{l} J_0(lr),$$

everywhere outside $r = 0$. The operator in the square brackets on the left hand side is the Laplacian operator $\nabla^2$ in cylindrical coordinates. It also easy to verify that

$$\lim_{\varepsilon \to 0} \int_0^\infty dl e^{-i\varepsilon} J_0(lr) = \lim_{\varepsilon \to 0} \frac{\varepsilon}{(r^2 + \varepsilon^2)^{1/2}} = 0.$$  

Integrating (A.3) with respect to $l$ from zero to infinity we see that the right hand side vanishes identically, therefore the function

$$\lim_{\varepsilon \to 0} \int_0^\infty dl e^{-i\varepsilon} g_1(z, z_1)J_0(lr)$$

satisfies identically the Laplace equation outside $r = 0$. Shifting the origin of the $x, y$ plane by an arbitrary vector $r_1$, changes nothing but the form of the distance from the origin of the plane to an arbitrary point: $r \to |r - r_1|$. As a result, the function

$$G(z, r, z_1, r_1) = \lim_{\varepsilon \to 0} \int_0^\infty dl e^{-i\varepsilon} g_1(z, z_1)J_0(l|r - r_1|)$$

is a solution of the Laplace equation at all points outside $r_1$. Throughout the paper the integral operator $\lim_{\varepsilon \to 0} \int_0^\infty dl e^{-i\varepsilon}$ will simply be denoted as $\int_0^\infty dl$.

Appendix B

The complete elliptic integral of the first kind (see [14], chapters 16 and 17)

$$K(x) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1 - x \sin^2 \phi}}$$

possesses the transformation property

$$K(x) = \frac{1}{\sqrt{1 - x}} K\left(\frac{x}{1 - x}\right)$$

and its asymptotic behavior near $x = 1$ reads

$$K(x) = \frac{1}{2} \log \frac{16}{1 - x}.$$ 

The transformation property allows us to derive the asymptotic behavior of $K(x)$ for $x \to -\infty$ which we shall need:

$$K(x) = \frac{1}{2\sqrt{-x}} \log(-16x).$$

Appendix C

We show here that the quantity

$$\int_{-c}^c \sqrt{c^2 - t^2} \frac{b'(t)}{z - t} dt$$

is a polynomial if the function $b(z)$ is a polynomial. (The integral is of course understood in the Cauchy principle value sense.) This is equivalent to showing that the quantity

$$\int_{-c}^c \sqrt{c^2 - t^2} \frac{t^k}{z - t} dt$$

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
is a polynomial for $k$ a natural number. The integral (C.2) can be re-written as
\[
\int_c^e \sqrt{c^2 - t^2} \frac{k^k - z^k}{z - t} dt + z^k \int_e^c \sqrt{c^2 - t^2} \frac{1}{z - t} dt.
\] (C.3)

The polynomial in the numerator of the first integral can factorized as $z - t$ times a polynomial in $z$ and $t$ of order $k - 1$. Therefore the first integral is non-singular and defines a polynomial of $z$ of order $k - 1$. The second integral (which appears in the chapter 5 of [14]) equals $c \pi z$. Therefore (C.2) is a polynomial in $z$ of order $k + 1$. Overall, if $b(z)$ is a polynomial then (C.1) is a polynomial (of the same order, as $b(z)$ enters (C.1) through its derivative).

Appendix D

The solution to the integral equation
\[
-2 \int_b^c \frac{f(z_1)}{z - z_1} dz_1 = \varphi(z)
\] (D.1)
for some function $\varphi(z)$, reads [13]
\[
-2\pi f(z) = -\frac{1}{\pi} \frac{1}{\sqrt{z - a} \sqrt{b - z}} \left[ C + \int_a^b \frac{1}{\sqrt{(u - a)(b - u)}} \varphi(u) du \right].
\] (D.2)

This is the analogue of Eq. (25) for the general interval $[a, b]$. Then, Eq. (30), which reads
\[
-2 \int_0^1 \frac{f(z_1) / \sqrt{t_1}}{t - t_1} dt_1 = \frac{b'(z)}{\sqrt{t}}
\] (D.3)

(where $z = 2c \sqrt{t} - c$ and $z_1 = 2c \sqrt{t_1} - c$) gives
\[
-2\pi f(z) / \sqrt{t} = -\frac{1}{\pi} \frac{1}{\sqrt{t(1-t)}} \left[ C + \int_0^1 \frac{1}{\sqrt{s(1-s)} \sqrt{2c \sqrt{s} - c}} \right].
\] (D.4)

which is precisely the solution given in Eq. (31).

Appendix E

The hydraulic potential $\Phi$ satisfies the Laplace equation for $r > 1$ and $-c < z < c$, and the boundary conditions (5b)–(5d). The boundary condition (5c) at ‘infinity’ is taken care of by the function $\Phi$ given in by Eq. (7), thus we only need to find a solution that satisfies (5a) and (5d) and vanishes at asymptotically large distances. We may solve the problem through Fourier series. We construct the solution associated with the linear case of the function $b(z)$ i.e. the monomial $z/c$, as explained in Section 3.2. As the $z$-dependent boundary data, i.e. the function $b(z)$, is an odd function, the solution will also be an odd function of $z$. We therefore expand in sine series of $z$. Then the boundary condition (5d) at $z = \pm c$ implies that we must expand in series of
\[
\sin \left[ \left( n + \frac{1}{2} \right) \frac{\pi c}{z} \right]
\] (E.1)
for $n = 0, 1, 2, \ldots$ (note that this index $n$ is merely the Fourier series counting index and has nothing to do with any other index $n$ appearing in the text).

Then the sought for solution $\Phi(r, z)$ is a superposition of functions
\[
X_n(r) \sin \left( \left( n + \frac{1}{2} \right) \frac{\pi c}{z} \right)
\] (E.2)
which satisfy the Laplace equation in for $r > 1$ and $-c < z < c$. The functions $X_n(r)$ are some kind of Bessel functions, which are known to arise in Laplace equation problems of cylindrical symmetry. Indeed the Laplace equation
\[
\left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2} \right\} \left( n + \frac{1}{2} \right) \frac{\pi c}{z} = 0
\] (E.3)
implies that $X_n(r)$ satisfy
\[
\left\{ \frac{1}{r} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} - \left( n + \frac{1}{2} \right)^2 (\pi/c)^2 \right\} X_n(r) = 0.
\] (E.4)

that is, they are related to the modified Bessel functions of the second kind $K_0(\chi)$ [13]: specifically $X_n(r)$ are proportional to

Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore rechare in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010
\[ K_0 \left( \left( n + \frac{1}{2} \right) \frac{r}{c} \right). \]  

That is the solution reads

\[ \Phi(r, z) = \sum_{n=0}^{\infty} c_n K_0 \left( \left( n + \frac{1}{2} \right) \frac{r}{c} \right) \sin \left( \left( n + \frac{1}{2} \right) \frac{z}{c} \right) \]  

for expansion coefficient \( c_n \) to be determined by applying the boundary condition (5b) at the wellbore wall:

\[ \sum_{n=0}^{\infty} c_n K_0 \left( \left( n + \frac{1}{2} \right) \frac{r}{c} \right) \sin \left( \left( n + \frac{1}{2} \right) \frac{z}{c} \right) = b(z) = \frac{z}{c}. \]  

The sine series are inverted by the orthogonality of the sine functions (E.1)

\[ \int_0^c \sin \left( \left( n + \frac{1}{2} \right) \frac{z}{c} \right) \sin \left( \left( m + \frac{1}{2} \right) \frac{z}{c} \right) dz = 0 \quad \text{for} \quad m \neq n, \quad \frac{c}{2} \quad \text{for} \quad m = n, \]  

through which we obtain

\[ c_n = (-1)^n \frac{2}{\pi^2 (n + \frac{1}{2})^2} K_0 \left( \left( n + \frac{1}{2} \right) \frac{r}{c} \right)^{-1}, \]  

which completes the construction of the solution.

References


Please cite this article in press as: C. Atkinson et al., On certain singular integral equations arising in the analysis of wellbore recharge in anisotropic formations, Appl. Math. Modell. (2015), http://dx.doi.org/10.1016/j.apm.2015.05.010